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Is the Hilbert space language too rich?

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Abstract

In order to answer this question, we analyse different phenomena occurring in general experimental set-ups arranged to analyse the properties of some unknown beams of particles. We arrive at the conclusion that sometimes the Hilbert space language appears to be too rich and also that there are some phenomena where the notion of transition probability disappears and any attempt to introduce it leads to the possibility of infinitely many inequivalent descriptions. Our analysis encouraged us to ask the question whether the Hilbert space language is not too rich in the more realistic situations, for example to deal with high-energy elementary particle scattering phenomena. A programme of investigations in that direction is formulated.

In the polemic with axiomatic quantum mechanics it is shown that the pure state concept can be formulated independently of the existence of any maximal filter.

1. *Introduction*

Axiomatic quantum mechanics aimed to prove the uniqueness of the quantum mechanical Hilbert space description for all future phenomena. The efforts were concentrated on a search for such a set of axioms, concerning the general structure of the propositions which can be said about the physical systems, which would imply the usual Hilbert space or algebraic representation.

The investigations started by Birkhoff & von Neumann (1936), and continued in many other papers (Dähn, 1968, Finkelstein, 1963; Finkelstein, Jauch, Schimonovich & Speiser, 1962, 1963; Gunson, 1967; Jauch, 1964, 1968; Jauch & Piron, 1963; Ludwig, 1967; Mackey, 1963; Mielnik, 1968, 1969; Piron, 1964; Pool, 1968a, b) led to different axiomatisation schemes with the required properties. Though some of the accepted axioms did not seem to be natural, the general belief is that the problem is solved and that we

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can freely use the usual Hilbert space language in future. One can claim that it is not true, because we have to deal with the rigged Hilbert spaces and because for the continuous spectrum the eigenvectors are only the distributions acting on some nuclear space ϕ, but in all practical cases we can use wave packets and regularised fields to obtain measurable results in the framework of some Hilbert space.

Axiomatic quantum mechanics was vigorously attacked by Mielnik (1968, 1969), who claimed to prove that the Hilbert space description is only one degenerate case of the infinitely many non-Hilbertian quantum worlds to be observed in the future. In this paper we show that such conclusions are not well. Mielnik's analysis is based on the quite unrealistic assumption that the physical transition probabilities between some pure states are equal to the static transmission probabilities between two maximal filters used for the preparation of these states.

In our opinion, the approach of axiomatic quantum mechanics is too general to give insight into some specific physical phenomena which can appear in different experimental set-ups. For this reason, we analyse the general experimental set-up E which can be used to investigate the phenomena characterising the ensembles of particle-beams. We assume that our experimental set-up E can consist of the following devices:

(i) the sources S which produce beams B;
(ii) the filters F which allow the division of beams into sub-beams having some common properties;
(iii) the transmitters T which change a beam b into a beam b_T;
(iv) the detectors D_p which register the intensity of the beams having a property p;
(v) the instruments I. A beam b enters into an instrument, the instrument measures some property, and a beam b_I goes out.

However, two observers investigating the same beam, but equipped with a different set of devices, can observe different phenomena and discover different mathematical schemes to describe them. Keeping this possibility in mind, we have been trying to analyse the different cases of the experimental set-up E, differing by the richness of the beams and the devices.

A careful analysis leads us to new definitions of filters, pure ensembles and to the important conclusion that in any considered case the Hilbert space description turns out to be possible. However, sometimes the data does not allow the extraction of the transition probabilities in a unique way, so it is more reasonable to abandon the Hilbert space description and to try to explain a causal evolution of the whole ensembles. Another feature which appears in our analysis is the fact that only some vectors and some scalar products in the Hilbert space description of the phenomena have a physical meaning; so, in some way, the Hilbert space language is too rich.

The 'too rich' language makes possible that using more or less phenomenological models we can always (in no unique way) explain the data without really broadening the understanding of them. The above-mentioned successes in the explanation of the data deepen the belief in the basic and unchangeable character of the language used and build a psychological barrier, making the discovery of a new, more economic and less ambiguous, language much more difficult.

All these considerations encouraged us to raise the important question whether the Hilbert space language is not too rich to explain the observed physical phenomena, for example in high-energy elementary-particle physics. A natural question arises: how could we find out whether this is the case? Although it is evident now that we cannot assign to all vectors in the Hilbert Fock space the physically realised states of the elementary particles system and that not all scalar products can be practically measured, yet it does not mean that the Hilbert space language is too rich. Similarly, in classical mechanics not every solution of an arbitrary Newton equation has a practical meaning and this does not mean that the language of classical physics is inappropriate.

To show that the Hilbert space language is too rich to deal with the scattering phenomena of elementary particles, we would have to show that, for example, the assumption of the unitary S matrix (which is derived using the assumption that any vector in the Hilbert state can be taken as an initial state) is violated. For example, we would have to find two such initial realisable states $|i_1\rangle$ and $|i_2\rangle$, which in our formalism must be represented by the orthogonal vectors, and show that the states $|Si_1\rangle$ and $|Si_2\rangle$ cannot be represented by the orthogonal vectors in the Hilbert space.

A careful analysis of these problems will be continued in the subsequent paper.

2. *General Experimental Set-ups*

At first we shall try to be as general as possible, so we shall consider two sets of objects: a set of sources and a set of devices. The interactions between beams, produced by the sources, and the devices give us the information about them which can enable us to make physics. In general, the information obtained is not unambiguous, so one has to accept some additional interpretational assumptions.

Usually one acts in a different way. Wanting to investigate beams, one constructs some devices based on the knowledge of classical and quantum physics. Such devices make possible the description of the unknown beams in terms of the quantities known before (like mass, momentum, energy, charge, spin, etc.). Such an approach is very reasonable, since it assumes the continuity of the science which worked so well before. However, let us cite Bohr (1961): 'The main point to realise is that a knowledge presents itself within a conceptual framework adapted to account for previous experience and that any such frame may prove too narrow to comprehend new experiences . . .' and '. . . when speaking of a conceptual framework we refer merely to the *unambiguous* logical representation of relations between experiences . . .'.

Keeping this in mind, we now forget about our science and we assume that we know nearly nothing about the sources and the devices. We want to investigate the problem of how we should deal with that case and what kind of language could be used to describe the observed phenomena. Let us start with some statements:

Statement 1. The sources S and all the devices used in the experimental set-up are given *a priori.* They should have the very important feature of reconstructability, by which we understand that identical set-ups can be constructed in any other laboratory at any time.

Statement 2. Among all the devices, we must have a counter g of quanta which must be used to find the intensity of the beams. This counter is at least one 'classical' device which is necessary to make quantitative 'quantum' physics. The problem is to construct such a counter for unknown beams; but let us assume that we have it. It need not be an absolute counter, like in Piron (1964), but it should be the most sensitive one available.

Statement 3. Using the counter g, we can observe the changes of the beam intensities after their interactions with the devices. Those interactions give us the information about the beams and the devices we have. This information allows us to classify the beams and to find among the devices such objects as filters, transmitters, etc. Of course, the information about the beams depends essentially on the devices used, and vice versa the information about the devices depends essentially on the beams which we have at our disposal. So everything we know is to a large extent relative, and we can never be sure that in the future we shall not discover other beams and devices which will change the interpretation of some old phenomena or which will force us to find a new theoretical language to describe the new ones. Similar views are contained in Mielnik (1968). In many cases it seems to be improbable, but it cannot be excluded.

Statement 4. Having some knowledge of the beams and devices, we have to choose some of them for further analysis. The chosen devices can be divided into two groups:

(a) preparatory and analysing devices,
(b) transmitters.

Such a division corresponds to three stages of the experiment in which they will be used:

(i) preparation,
(ii) transmission,
(iii) detection.

Statement 5. In the preparation stage we classify and prepare the beams. We introduce the concept of pure and mixed beams. Knowing the properties of the pure beams, we can ascribe states to them. Thus the preparatory stage enables us to find a set of initial states whose change in the transmission stage we should try to explain.

Statement 6. In the transmission stage we let our beams go through some chosen devices called transmitters. By transmitters we can understand also the action of the external fields. If the beam was under the influence of the external field for the time Δt before detection, we can say that it went through the transmitter $T(\Delta t)$. Thus we can have the approximately instantaneous change of the beam or we can observe its more or less continuous evolution.

Statement 7. After the transmission stage we classify obtained beams and we try to find some mathematical language and a model allowing the interpretation of the observed regularities.

Now we have to find out the meaning of some terms which appeared in these statements. It turns out that the definition of filters and pure states is not obvious. We cannot see the details of the transmission process as it looks inside a device. Therefore what we know is the change of intensity of the incoming beam.

Before starting a more detailed discussion, we must add some assumptions in the spirit of Statement 1. We have to work with an approximately stable source, since in order to make predictions we have to know the intensities of the beams relying on previous measurements. We must also assume that our devices have no memory and act in the 'same' way on the 'same' beams. So in fact we are always dealing with ensembles $\tilde{b}$ of the identically prepared beams b. Performing many experiments we find the properties of the ensembles $\tilde{b}$ and often we can ascribe them to every beam of the ensembles $\tilde{b}$. If it is possible, we shall talk about the beams and their properties instead of talking about the ensembles. In our case we cannot always prepare arbitrary mixtures of the produced beams. First we have to check whether the beams behave in a 'classical' or in a 'quantum' way. We now make a short digression about such behaviour, which will be a short repetition of well-known things.

If the beams and devices behave classically, then each quantum of the beam can be characterised by some properties, possessed in an attributive way, which can be found with the help of measurements. These measurements can by no means change the properties of the quanta. A device d which is transparent only to the quanta having a property 'd' is called a classical filter. Such a device is of course idempotent, which means that it is neutral to all quanta to which it is transparent. If the quanta also have some other properties, we can construct, in principle, maximal filters–transparent only to the quanta having all properties the same. Pure beams are those which go through the maximal filters without change.

In quantum physics, a quantum can have a property with certainty, but only up to the moment of the measurement of another property which is incompatible with the first one. To discover the quantum behaviour one must show the incompatibility of some properties. For this purpose we must find at least two idempotent and incompatible devices d and l to perform the following experiment with a beam b. We transmit the beam b through the device d and obtain a beam b_d, for which a device d is transparent. Now we transmit the beam b_d through the device l and we obtain a beam b_{dl} of smaller intensity. Now the device l is transparent to the beam b_{dl}. Finally, we transmit the beam b_{dl} through the device d, obtaining a beam b_{dld}. If the beam $b_{dld} \neq b_{dl}$, then we can say that the beams and devices do not behave in a classical way.

If, on the other hand, $b_{dld} = b_{dl} = b_{ld} = b_{dldl}$ and other devices incompatible with d and l cannot be found in our experimental set-up, then we can say that the beam b_{dl} is pure, the devices l, d and $l \cdot d$ are classical filters and the device $l \cdot d$ is a maximal filter in our set (for simplicity we exclude the existence of other compatible more restrictive filters). Each quantum of the beam b_{dl} has two properties 'd' and 'l'—the filters d and l are transparent to it.

In the quantum case our classical picture of a filter has to be completely changed. We cannot say, as in Piron (1964), '. . . quantum mechanical filter selects single particle properties'. All the quanta of a beam b_d have the same property 'd' (the device d is to them transparent), but after going through the device l not all of them can still have a property 'd'. So the idempotent device l does not select the quanta having a property 'l' but it only transforms with a probability $p(d, l)$ some quanta having a property 'd' into a quanta having a property 'l' and absorbs ones which are not transformed.

We cannot explain this probabilistic approach assuming that the beam b_d is a mixture of the quanta labelled by hidden parameters ξ *constant in time* and that the device l is a classical transmitter which can change the property 'd' and the parameters ξ in a well-defined causal way depending on their initial values. The non-existence of hidden parameters of this kind was shown in a different formalised language by Jauch & Piron (1963). So we have to assume that the devices l and d act in an intrinsically probabilistic way, but now we can ask whether it is possible to check that the beam b_d is a pure beam.

Let us, for example, assume that the beam b_d of average intensity I is a mixture of two beams of intensities I_1 and I_2 consisting of quanta A and B, respectively. Let the device l act in the following way:

it transmits each quantum A with a probability a and changes it into a quantum C;
it transmits each quantum B with a probability b and changes it into a quantum D;
it transmits all the quanta D and C without change.

Let the device d act in a similar way:

it transmits each quantum C with a probability c and changes it into a quantum A;
it transmits each quantum D with a probability k and changes it into a quantum B;
it transmits without change all quanta A and B.

The transmission probabilities $p(d, l)$ are strictly defined in the following way

$$p(d, l) = Ss(r)r(d, l)\, dr \tag{2.1}$$

where S denotes a sum or an integral over all values of $r(d, l)$; $s(r)$ is normalised to the unity probability distribution of the ratios $r(d, l)$; the ratios $r(d, l) = I_l/I_d$ where I_d and I_l are the intensities of the beams b_d and b_{dl} for all beams $b \in \tilde{b}$. All other probabilities met later are defined in a similar way. The probabilities $p(d, l)$ and $p(l, d)$ must be the same for all pairs of *d-l* and *l-d*, respectively, occurring in the chain *d-l-d-l-d-l* of the experiments with the ensemble $\tilde{b}$. It gives us the constraints on the possible values of a, b, c, k. Other constraints are obvious: $0 < I_1, I_2 < I, I_1 + I_2 = I, 0 < a, b, c, k < 1$. If we analyse these constraints we come to the following corollary:

Corollary. The probabilities a, b, c, k must satisfy the following condition $a \cdot c = b \cdot K = w$; then $p_\lambda(d, l) = (a + b \cdot \lambda)/(1 + \lambda)$ and $p_\lambda(l, d) = w/p_\lambda(d, l)$ where $\lambda = I_2/I_1$. So for every two experimental numbers $p_\lambda(d, l)$ and $p_\lambda(l, d)$ we can adjust two parameters to make the above interpretation possible.

However, as we see $p_\lambda(d, l)$ depends on the relative intensity λ of the two hypothetical sub-beams; so we can verify our hypothesis by trying to change λ in the beam b_d. We do not have any other way than to cause the decrease of the intensity of the beams $b_d \in \tilde{b_d}$ by different methods. If we do not obtain different values for $p_\lambda(d, l)$ and $p_\lambda(l, d)$, then we must reject our hypothesis and state that it is not legitimate to assume that the beams b_d consist of sub-beams, so they can be called pure. But by the expression 'a pure beam' we should not understand a beam consisting of identical quanta, since the term 'identical' is classical and means 'behaving in the same manner in all situations'. The devices l and d do not treat all the quanta from the beams b_d and b in the same way. We cannot understand the mechanism of this differentiation and also usually we do not observe separate stages of the transition. We just observe the behaviour of the beam b_d as a whole and find the statistical regularities. Therefore, in the theoretical analysis of the process we should not represent the transmission of the beam b_d by a set of yes–no experiments with each quantum, but more properly we should talk about the properties of the beams as a whole and about states of the beam instead of talking about the states of the single quanta. In some situations it can happen that only the states of ensembles $\tilde{b}$ have a precise meaning. We shall discuss such situations later. This wholeness of the physical phenomenon in the microworld was wisely pointed out by Bohr (1961) many times.

We have spent so much time discussing the devices l and d because they behave in a way analogous to the behaviour of the tourmaline plates which are usually called filters. We wanted to show to what extent they are not classical, if discussed in terms of corpuscular language. Their filtering properties can be understood in the language associating a wave to each beam. We also wanted to get an intuition enabling us to define filters and pure beams in our poor information system.

Definition 1. Filters are devices which:

(i) are idempotent;
(ii) for all beams $b \in B$ entering an arbitrary chain consisting of the filters $f_1, f_2, \ldots$, the transmission probabilities for each pair f_i–f_j are constant; $p_b(f_i, f_j) = \text{const}_1(b), p_b(f_j, f_i) = \text{const}_2(b)$;
(iii) from all devices D satisfying the conditions (i) and (ii), for each device d we can find a set 0_d consisting of all devices l_j which have the same

transmission probabilities to all other devices from D as a device d has, namely

$0_d = \{l_j \in D; p_b(l_j, d_i) = p_b(d, d_i)$ for all $b \in B$ and all $d_i \in D\}$

A filter is a minimally transparent element in the set 0_d. This means that $I_{bf} \leqslant I_{bl_j}$ for all $l_j \in 0_d$ and all $b \in B$. If the minimally transparent element in 0_d cannot be found, then we call all the devices 0_d relative filters and to further analysis of the beams we choose one of them.

The long property (iii) enables us to differentiate between classical filters and some similar classical transmitters. Our definition of the filters is different from that given, for example, by Mielnik (1969), and many relative filters from his work are treated like normal filters, as they should be.

Definition 2. Provisory maximal filters are the minimally transparent elements in maximal sets of the compatible filters.

Definition 3. A provisory pure beam is one for which the provisory maximal filter is transparent. We use the term 'provisory' since we are not sure whether the set of filters which we have in E is a maximal one. In the transmission stage some provisory pure beams can behave in a way suggesting that they consist of the two sub-beams not separated in the preparation stage.

Usually it is assumed that two filters d and l are characterised by the transmission probability independent on b. In this paper we assume that the beams can be characterised by many properties and the same filters can be sensitive on different properties in a different way. For example, the filter d can be transparent to all beams having a property 'p_1' but reduce the intensity of the beams having a property 'p_2'. If b_1 denotes a beam with the property 'p_1' only, and b_2 denotes a beam with the property 'p_2' only, then it can happen that $p_{b_1}(d, l) \neq p_{b_2}(d, l)$.

Besides the transmission probabilities $p_b(f_i, f_j)$, which we shall denote in all practical cases $p(f_i, f_j)$, we also need the filtration probabilities

$$p(b, f_i) = Ss(r)r(b, f_i)\, dr \tag{2.2}$$

where a filtration ratio $r(b, f_i) = I_{fi}/I_b$ with I_{fi} and I_b denote the intensities of the beams b_{fi} and b respectively.

Concluding, if we have the filters in E then we can find provisory pure beams and investigate only their properties. If we do not have the filters, we must have some other devices for the determination of the initial states. Such devices are the detectors D_p mentioned in the introduction. From all counters of quanta which we have in E besides the counter g from Statement 2 we eliminate all those which overlook some quanta independent of their properties. They can be recognised by the proportional decrease of the registered intensities of all the beams. We also eliminate all non-linear counters. All those remaining are called D_p. Now with each beam b we can associate registration probabilities by all the detectors D_p. A registration probability $p(b, d)$ is defined by the detector d as

$$p(b, d) = Ss(r)r(b, d)\, dr \tag{2.3}$$

where registration ratios $r(b, d) = I_d/I_b$ with I_b and I_d denoting the intensities of the beam b measured by the detectors g and d, respectively. (From this moment the detectors will be denoted solely by d_i and the filters by f_i to differentiate between the two kinds of probabilities $p(b, f_i)$ and $p(b, d_i)$.)

If the probability $p(b, d) = K$, we can say that an average quantum of the beam b has the property 'd' (to be registered by the detector d) with the probability K. As usual, we must check the character of the observed probabilities using different intensity reduction procedures.

To visualise what kind of effects can appear in the case discussed above, we shall consider a simple example.

Example. Let us consider four beams of classical objects produced by four sources, i.e., the beams of balls in three colours: pink, green and blue. All the

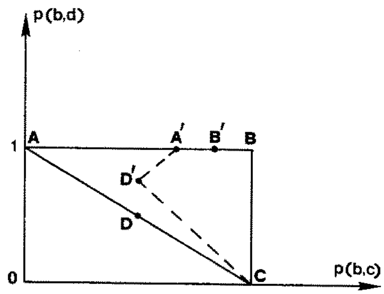

Figure 1.

balls behave in an identical way in all macroscopic experiments. So from the point of view of a colour-blind observer they are identical. However, the observer has three additional detectors: g, d and c; g registers all balls, d registers all pink and green balls, and c registers all green and blue balls.

After repeated experiments, the observer notices that each beam b is characterised by the two registration probabilities $p(b, d)$ and $p(b, c)$, defined as before. The observer checks the stability of the values of $p(b, c)$ and $p(b, d)$ by stopping some of the balls before they arrive at the detectors. Of course, he discovers that the beams behave like classical mixtures, but being unable to select the pure beams he can only represent the beams by some points in two-dimensional vector space. If all possible mixtures of the initial beams can be experimentally realised, then all these mixtures can be represented by a convex set on a plane. The specific shape of this set depends on the initial beams. One can say that each set is a convex envelope of the set of the points corresponding to the initial beams. Let us visualise this in a simple picture (see Fig. 1).

The triangle ABC in Fig. 1 is a classical symplectic cone (Mielnik, 1969); the observer notices that the beams A, B, C are pure and the beam D is a mixture of them.

Every ball of A has a feature 'd' and does not have a feature 'c'.
Every ball of B has both features 'd' and 'c'.
Every ball of C has a feature 'c' and does not have a feature 'd'.

The quadrangle C, D', A', B' is another set of the initial beams; now only the beam C is pure and the other beams are mixed, but since we cannot separate pure beams we can investigate the behaviour of all C, D', A', B' beams in the transmission stage.

The possibility of representing all the states by all transition probabilities was pointed out (in a different context) by Haag & Kastler (1964). They also stressed that we know the transition probabilities only approximately, due to the experimental errors and limited precision of the instruments. However, the beams A', B', C, D' are represented by well-separated points in the two-dimensional vector space, so we are not afraid of ambiguities.

If, in some other experimental situation, we obtained the same set A', B', C, D' and the beams showed quantum character, then we would assume that the beams A', B', C, D' are pure but we would represent them in the same way.

Now we want to investigate the behaviour of our beams in the transmission stage.

Definition 4. A classical transmitter T is a device which changes the beam b in a unique way into a beam b_T and which is not a filter. A quantum transmitter T is a device which changes the pure beam b into the beams b_s with fixed transition probabilities $p_T(b, b_s)$ and which is not a filter.

Coming back to our example (classical case), we take as a transmitter T the device which changes the colours of the balls in a well-prescribed way. For example, it can change the beam b consisting of green and pink balls into the beam b_T consisting of balls in one or two other colours. The classical transmitter of this type has a characteristic feature of repeatability: in the chains $b, b_T, b_{TT}, \ldots$ cycles must appear. The experimental values of the registration ratios $r(b_T, c)$ and $r(b_T, d)$ form sharp one-peaked probability distributions $s(r)$, enabling the easy calculation of the registration probabilities $p(b_T, c)$ and $p(b_T, d)$.

If we have a quantum transmitter T' and a quantum beam b, there is no reason for the above-mentioned repeatability. Also, if the transmitter T' transforms the beam b into a set of well-separated beams b_s with the transition probabilities $p_T(b, b_s)$, then the observed experimental values of the registration ratios $r(b_{T'}, c)$ and $r(b_{T'}, d)$ (at least one of them) should form many-peaked probability distributions $s(r)$ with sharp peak values around $r(b_{T'}, c) = p(b_s, c)$ and $r(b_{T'}, d) = p(b_s, d)$, respectively. Therefore, analysing the distributions $s(r)$, one can (in principle in this case) determine the beams b_s and the transition probabilities $p_T(b, b_s)$ uniquely (at least if all $p_T(b, b_s)$ are different).

Wanting to represent mathematically the transmitters T and T', we easily find that T can be represented by a matrix whose range of the domain $A'B'CD'$ must be a convex subset of the square $OABC$. The beam b_T can be represented by a vector $\mathbf{b}_T = (p(b_T, c), p(b_T, d))$. The registration probabilities can be obtained as scalar products with the vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, respectively. In the quantum case, to each beam one can only associate a probability measure on the square $OABC$. Then T' transforms measures μ_b (which are nearly 1 on the vectors $\mathbf{b}$ and go quickly to zero outside) into measures $\mu_{bT} = \Sigma_s \, p_T(b, b_s)\mu_{b_s}$.

Remark 1. However, it can happen that the distributions $s(r)$ cannot be interpreted in a unique way by means of the transition probabilities $p_T(b, b_s)$. It can even turn out that they can be interpreted in infinitely many ways. This leads us to a serious revision of the definitions of pure beams and transmitters. As we know only the probability distributions $s(r)$ characterise the ensemble $\tilde{b}_{T'}$ completely, the values of the registration probabilities $p(b_{T'}, c)$ and $p(b_{T'}, d)$ can characterise the ensemble $\tilde{b}_{T'}$ well only if the distributions $s(r)$ are sharp, symmetric and one-peaked distributions. In such a case, having a two-number classification of the ensemble $\tilde{b}_T$, we can ascribe the same numbers (with the experimental errors) to all member beams $b_{T'} \in \tilde{b}_{T'}$, and even to the average quanta from each beam $b_{T'}$. If the probability distributions $s(r)$ have a reach structure, then they only characterise the ensemble $\tilde{b}_{T'}$ adequately and we can only discuss well-defined states of the ensemble $\tilde{b}_T$. This consideration leads us to the following definitions.

Definition 5. A state of the ensemble $\tilde{b}$ can be completely characterised only by the probability distributions $s(r)$ of the filtration or the registration ratios for all beams $b \in \tilde{b}$.

Definition 6. A pure ensemble $\tilde{b}$ of pure beams b is characterised by such probability distributions $s(r)$ which remain approximately unchanged:

(i) for the new ensembles $\tilde{b}_i$ obtained from the ensemble $\tilde{b}$ by the application of the ith intensity reduction procedure on each beam $b \in \tilde{b}$;
(ii) for all rich sub-ensembles of $\tilde{b}$ chosen in a random way.

Definition 7. A transmitter T is a device which transforms each ensemble $\tilde{b}$ into a well-defined ensemble $\tilde{b}_T$ and which is not a filter.

Now we come to the general conclusions of this long section.

Conclusions

We have considered the experimental set-ups with and without provisory maximal filters (p.m. filters). We also divide into two parts the discussion of mathematical schemes useful for the representation of the results.

A. We have n-p.m. filters $f_1, \ldots, f_n$, which we use not only for the preparation of p.p. beams but also to analyse the final beams obtained in the transmission stage. Now we can have two cases:

(a) In the detection stage we always observe the p.p. beams, but starting from the same p.p. beam b and using the same transmitter T we observe different outgoing pure beams b_i. However, they appear with more or less fixed transition probabilities $p_T(b, b_i)$. In such a situation, we can represent the beams b_i and b_T by the vectors in the n-dimensional euclidean space E_1. The initial beams b_i can be represented by the basic vectors $\mathbf{e}_i$ and the beams b_T by the vector $\mathbf{b}_T$ in this space. The probabilities of finding the beams b_i as the outgoing beam b_T can be written in a scalar product form $p_T(b, b_i) = \mathbf{b}_T \cdot \mathbf{e}_i$. Each transmitter T can be represented by a linear operator acting in this space.
(b) The beam b_T never turns out to be one of the p.p. beams b_i. We obtain only probability distributions $s(r)$ of the filtration ratios $r(b_T, f_i)$,

which in general have rich peak structure. As we noted in the example, it is possible to describe the ensemble $\tilde{b}_T$ of the beams b_T by probability measures μ_T on the n-dimensional vector space E_2. The vectors $\mathbf{b}$ in E_2, corresponding to beams b, have as their components the filtration probabilities $p(b, f_j)$. The measures μ_b, corresponding to the initial p.p. beams b_i, are nearly equal to 1 on the corresponding vectors $\mathbf{b}_i$ and go quickly to 0 outside. The measures μ_{bT} are only characterised by all $s(r)$. Only in some cases of the $s(r)$ we can extract the transition probabilities $p_T(b, b_s)$ in a more or less unique way. The beams b_s are different from the initial beams b_i. The transition probabilities satisfy a condition $\Sigma_s\, p_T(b, b_s) = 1$, contrary to the fact that for any chosen beam b_T from the ensemble $\tilde{b}_T$ in general $\Sigma_i\, r(b_T, f_i) \neq 1$. In other cases we can fit our experimental data on $s(r)$ with the different $p_T(b, b_s)$ in many different ways. To have uniqueness we have to accept that the ensemble $\tilde{b}_T$ is only characterised by all $s(r)$, as was already stated in Remark 1 and in Definition 5. In such an approach there is no place for the notion of transition probabilities.

B. The case when we have only n-detectors $d_1, \ldots, d_n$ is mathematically equivalent to the case A(b). We have only to replace the filtration ratios and probabilities by the registration ratios and probabilities, respectively.

In all cases we should check the purity of the ensemble b_T according to Definition 6. Sometimes we can interpret the values of $p_T(b, b_s)$ found, due to the mixed character of the initial ensemble $\tilde{b}$ with respect to the property analysed by the transmitter T. Such a possibility explains the term 'provisory' occurring in the definitions of the maximal filter and the pure beam. A contrary situation is also possible. Mixed ensembles with respect to some properties can behave like pure ones with respect to the properties analysed by the transmitter T.

The careful differentiation between filtration or registration ratios and the transition probabilities leads us to the conclusion that, in experiments of the type considered, if we succeed with the extraction of the transition probabilities from the experimental data, then in all cases A and B we can represent the beams by unit vectors in the n-dimensional Hilbert space for A(b) and B and the transition probabilities by the appropriate scalar products between them. It stems from the fact that for each ensemble $\tilde{b}_T$ we have a finite number of beams b_s and the transition probabilities $p(b_T, b_s)$, which can be embedded in the Hilbert space. However, the following new features appear:

(i) The set of physically meaningful vectors is restricted to those corresponding to the b_i (initial beams), to all beams $b_{iT_i}, b_{iT_iT_j} \ldots$ (where T_i are all available transmitters), and also to all beams b_s extracted in the analysis of the data. If we have two vectors $\mathbf{b}_{T_1}$ and $\mathbf{b}_{T_2}$ we do not know whether $\mathbf{b}_{T_1} + \mathbf{b}_{T_2}$ corresponds to a physically realisable beam b_{T_3}.

(ii) The physical interpretations have only some scalar products of the vectors of the type $|\langle \mathbf{b}_T | \mathbf{b}_s \rangle|^2 = p_T(b, b_s)$, where b_s are found in the analysis of all $s(r)$ describing $\tilde{b}_T$ and $\mathbf{b}_s$ are vectors corresponding to them. All the $\mathbf{b}_s$ found for all transmission processes form an orthonormal basis of the Hilbert space considered.

If we accept the philosophy of Definitions 5–7, then the Hilbert space description loses its sense, since the transition probabilities $p_T(b, b_s)$ do not appear.

However, in all the cases discussed above, having many experimental results we can try to find a quasi-theory enabling us to interpret the data and to predict new results.

So we find that the usual quantum mechanical description can turn out to be not too poor, as was suggested by Mielnik (1968, 1969), but too rich or not very appropriate.

The experimental scheme discussed so far is structurally similar to that used for the investigation of the scattering phenomena of elementary particles. Instead of simple filters and detectors we use many complicated instruments which, based on our previous knowledge, enable us to prepare and classify the initial and final beams. The different kinds of chambers and emulsion layers enable us to observe one-particle beams; but, in fact, we observe only statistical regularities characterising ensembles of such beams. The scattering process can be understood as specific transmission processes in two ways. One interpretation is that one-particle beams, for example proton beams b_p, are transmitted by the transmitters = protons T_p leading to a many-particle beam $(b_p)_{T_p} = b^f_{p\text{-}p}$. Another interpretation is that initial beams $b^i_{p\text{-}p}$ consisting of the two free protons are transmitted by the transmitters = strong p–p interaction into the final beams $b^f_{p\text{-}p}$. We still have an additional interpretational freedom; by a transformed ensemble $b^f_{p\text{-}p}$ we can understand a set of all final free particle beams $b^f_{p\text{-}p}$ or a set of strongly interacting proton–proton beams $b^s_{p\text{-}p}$, which are visualised by the interaction points in the emulsion layers or the photo-pictures from the different chambers. In this second case, the final free particle beams $b^f_{p\text{-}p}$ can be interpreted as arising in some kind of measurement process performed on the beams $b^s_{p\text{-}p}$.

One could argue that the formalism discussed in this paper is not very applicable, since in elementary-particle physics we deal with the continuous variables characterising the beams. However, in all preliminary experimental data we characterise our initial and final beams by intervals of, in fact, continuous variables. All these analogies, and the fact that a set of the different scattering processes which can be observed between elementary particles is very limited, show the need for careful investigation of whether the quantum mechanical unitary S-matrix language is not too rich for the description of the observed phenomena. We shall investigate this problem in detail in the subsequent paper.

Now we give for completeness a definition of instruments, leaving the analysis of the practically realised instruments to the subsequent paper.

Definition 8. An instrument I is a device allowing the description of each beam b in terms of earlier known categories (parameters). The ascription of these parameters involves the assumption of the applicability of some earlier known theories. The instrument, by its interaction with the beam b, changes it into a beam b_I. The repeated application of the same instruments usually leads to slightly different values of parameters ascribed to our beams. The measurement made by the instrument for all beams b characterises the ensemble $\tilde{b}$. An ideal instrument is such that we can assume that $b_I = b$.

Generally, one could consider the instrument which can change the ensemble $\tilde{b}$ into the different ensemble $\tilde{b}_I \neq \tilde{b}$. Analysing the results of many measurements, one could find the characteristic features of the instruments used. In spite of the fact that $\tilde{b}_I$ is different from $\tilde{b}$, the values of parameters ascribed to $\tilde{b}$ in the measurements can be used in some way for the labelling of the initial ensembles $\tilde{b}$ or $\tilde{b}_I$. However, in practice we try to use the ideal instruments. A good example of such instruments are the filters f_i as applied on the beams b_{f_i}, different kinds of chambers and so on.

At the end of this section we should like to point out that, in spite of the fact that we have been talking about ensembles of quanta beams, our results can be generalised for experiments with ensembles of identical physical systems. Instead of filters and detectors, we should have other more complicated instruments to determine the states of the initial and final ensembles.

We should also like to remark that the particle character of our beams, implying measurement of the beam intensities by counting the quanta, is not necessary. When discussing states of ensembles $\tilde{b}$, we can measure the intensities and the appropriate ratios for some non-particle beams. The particle character of the beams was essential in the discussion of the pure beams and of the non-classical character of the probabilities.

Now we pass to the polemic with some views presented in papers on axiomatic quantum mechanics.

3. *Polemic with Axiomatic Quantum Mechanics*

The great success of quantum mechanics in describing many atomic and sub-atomic phenomena, and the fact that classical physics is a limiting case of quantum physics, encouraged people to think that a general framework to describe all physical phenomena had been discovered.

To prove this statement one should find such a set of the natural assumptions on S, B, F, D, T and I characterising a physical process of a measurement in general, which would imply uniquely the usual quantum description.

The more general attitude was accepted in the papers of the Birkhoff & von Neumann (1936), Mackey (1963), Jauch (1964, 1968), Jauch & Piron (1963), Piron (1964), Finkelstein (1963), Finkelstein, Jauch, Schimonovich & Speiser (1962, 1963), Gunson (1967), and others, where the so-called quantum logic of the propositions concerning a physical system was studied. In the papers of Ludwig (1967, 1968) and Dähn (1968) the state-effect structures were investigated; in the papers of Pool (1968a, b) state-event structures.

All these studies aimed to find such a set of natural axioms which would imply uniquely the use of the complex Hilbert space language or the algebraic Haag-Kastler (1964) language for the description of the states and transition probabilities. The required set of assumptions was found in many axiomatic approaches; however, the naturality of some of the accepted assumptions is questionable. They were all chosen by analogy to the experiments performed on the optical bench with the use of colour filters, Nicol prisms, and other devices. The states of the differently polarised light can be represented by all rays in the complex two-dimensional Hilbert space $\mathscr{H}(2, C)$ and each state can be realised in the laboratory. To each linearly polarised beam there corresponds in a one-to-one way an appropriate filter–the Nicol prism or polarisation filter–which is transparent to it. So one has, in principle, an uncountable amount of filters in the laboratory, since in that case nearly each rotation performed on the Nicol prism enables its interpretation as a different filter.

This special case strongly supports the commonly accepted philosophy in quantum mechanics, according to which each pure beam state is prepared by an appropriate maximal filter. Also the filtration probabilities of the pure beams (2.2) in that case are equal to the transmission probabilities (2.1) between the corresponding filters and can be expressed by the scalar products of the unit vectors in $\mathscr{H}(2, C)$ representing those filters. So in fact, instead of talking about the states, one can talk about the filters and the transmission probabilities between them.

This observation gave Mielnik the force to attack the usual quantum logic approach. His main starting point (Mielnik, 1968) was assumption that the set of filters with the geometry implied by the transmission probabilities is the main characteristic of all quantum phenomena (however, Mielnik, instead of saying 'transmission', says 'transition'. Therefore, to investigate the problem of the universality of the orthodox Hilbert space representation, one must study whether the above-mentioned geometry allows the representation of the filters by the unit vectors and the transmission probabilities as scalar products. In his two clear and provocative papers (1968, 1969), he realises his programme and comes to the following conclusions (1969):

> '. . . It now becomes clear that the orthodox classical and orthodox quantum systems do not represent a unique alternative for quantum theories, but they are only particularly degenerate members of a vast family of 'quantum worlds' which are mathematically possible . . .'
>
> '. . . We thus conclude that the concepts reviewed in this article represent the missing element necessary to convert non-linear wave mechanics into 'mechanics of non-linear quanta . . .'.'

Though there is no mathematical fault in the papers (Mielnik, 1968, 1969), in our opinion the above statements are not well justified. A simple misunderstanding is due to the interpretation of the transmission probabilities as the transition probabilities. These latter are a basic notion measured in all our experiments and depending on the dynamics of the phenomena. The transition probabilities can be directly connected with the cross-section, branching-ratios, life-times of the excited levels, and so on. The value of quantum mechanics consists in its ability to predict those probabilities in the agreement with the experimental data. On the other hand, the transmission probabilities are the static properties of the filters and the beams and can only be used (if the filters exist) to characterise the initial and final beams. A careful analysis of the general experimental set-ups made in the previous section allowed a clean differentiation between all kinds of probabilities [(2.1), (2.2), (2.3)] and Definition 6 divorced the concept of a pure state with a concept of a maximal filter. So the Mielnik statement has to be changed into the following statement:

There can be many sets of filters whose transmission probabilities do not allow the representation of them by the unit vectors in the Hilbert space with the transmission probabilities being equal to the appropriate scalar products.

Besides this main criticism, we have other critical remarks concerning the paper (Mielnik, 1969). In this paper the maximal transmission systems are considered. Those systems have so rich a class of transmitters that each physical state can be transformed in any other by means of the appropriate transmitter. They also have, in general, an uncountably rich set of maximal filters. To each pure beam there correspond two filters; one is completely transparent to the beam, the second is completely non-transparent. In our opinion, dealing with such rich classes of filters is rather unrealistic. Therefore, we cannot accept the second-cited conclusion concerning the quantisation of non-linear theories. The procedure proposed in the paper (Mielnik, 1969) can be devoid of any physical meaning.

To illustrate our arguments we shall discuss a nice example of the drop of non-Hilbertian quantum liquid from Mielnik's (1968) paper:

> '. . . Someone looked at a small spherical glass bubble: inside there was a drop of liquid. The drop occupied exactly half of the bubble in the shape of a hemi-sphere. He was able to introduce inside a thin, flat partition dividing the interior of the bubble into two equal volumes. He tried to do this so that the drop would become split. However, the drop exhibited a quantum behaviour: instead of being divided into two parts, the drop jumped and occupied the space on only one side of the partition. He repeated the attempt, obtaining a similar result. He began to observe this phenomenon and discovered that each time the partition is introduced the drop chooses a certain side with a definite probability. This probability depends upon the angle between the partition and the initial surface of the drop. If the drop occupied a hemisphere s and the partition forces it to choose between the two hemispheres r and r', the probabilities of transition into r and r' are proportional to volumes of $s \cap r$ and $s \cap r'$. He was struck by the analogy between positions of the drop and quantum states and between the partition and the macroscopic measuring apparatus. He wanted to formulate the quantum theory of this phenomenon, but he realised that he could not use Hilbert spaces because the space of states of the drop was not Hilbertian . . .'.

We disagree with that conclusion and we analyse the behaviour of the observer. To make some predictions he considers an ensemble $\tilde{b}$ of the above-mentioned bubbles b. Before starting to divide the liquid drop he fixes the positions of all the bubbles to make the surfaces of all the drops horizontal. He chooses a well-separated set of N partitions t_i labelled by the angles α_i between partitions and the surface of the drops. After many partitions have been made, he observes $2N$ possible positions of the drops after partitions. Repeating the experiments with fixed partition t of the ensemble b he finds out the probabilistic behaviour of the drops with some fixed transition probabilities to the final states. Before accepting a usual quantum interpretation of the probabilities he investigates the purity of the ensemble $\tilde{b}$ as described in Definition 6. If the ensemble $\tilde{b}$ turns out to be pure, he accepts the following interpretation. The partition t is some kind of interaction exerted on the drops so the partition t is some kind of quantum transmitter transforming the ensemble $\tilde{b}$ into pure ensembles $\tilde{b}_1$ and $\tilde{b}_2$ with fixed transition probabilities, $p_t(b, b_1)$ and $p_t(b, b_2)$. Of course, those probabilities for all the partitions t_i we can represent as a scalar products in $2N$ dimensional real Hilbert space of the vectors $\mathbf{b}_{ti}$ corresponding to the b_{ti} with appropriate vectors $\mathbf{b}_i$'s. Naturally, the scalar products $\mathbf{b}_{t_1} \cdot \mathbf{b}_{t_2}$ have no physical meaning. Therefore, the Hilbert space description of this phenomenon is in some sense too rich and not too poor, as was claimed in Mielnik (1968).

Returning to the discussion of axiomatic quantum mechanics, we state that in our opinion the problem of Birkhoff and von Neumann, although skillfully solved in the different axiomatisation schemes, was stated in too general a way. In our opinion, it is not very economic to talk about all possible propositions concerning the physical systems in general. In practice we have to perform the experiments and the analysis of the results gives us a set of physically meaningful propositions about the system. This set depends on the particular experimental set-up and its richness depends on the richness of the observed phenomena. The careful analysis of the particular experimental set-ups can lead us to the discovery of new, more economical and fruitful descriptions, though the old language of Hilbert spaces could be used. Being too general, we cannot get insight into such problems and we cannot hope to arrive at the conclusive new statements to be verified in the experiments.

Finally, we should like to question some axioms of Gunsón (1967) and Pool (1968a). Gunson considers a set of propositions P and a set of states S. States are the probability measures on the propositions, taking the real values from 0 to 1. The axiom A.4 is: 'For every $a, b \in P$ we have $a \leqq b$ if and only if $f(a) \leqslant f(b)$ for all $f \in S$. For the propositions, the relation $a \leqq b$ is equivalent to the usual implication relation a implies b. Gunson also uses the following definition of the orthogonality $a \perp b \leftrightarrow a \leqq b'$, where b' is the logical negation of the proposition b.

Counter example. Let us consider the following situation. We have only two pure ensembles $\tilde{f}$ and $\tilde{g}$ and two detectors d and l. The only things we can measure are the registration probabilities (2.3) by those detectors. With each detector we can associate two propositions. For example: the proposition 'd'–'the physical system from the ensemble is registered by the detector d', and the proposition 'd'''–'the physical system from the ensemble is not registered by the detector d'. As we see for $f \in \tilde{f}$ we have $p(f, d) = f(\text{'}d\text{'})$ in Gunson's notation.

Now let us assume that we observe the following values of $p(f, d)$ and $p(f, l)$: $f(\text{'}d\text{'}) = 1/4$, $g(\text{'}d\text{'}) = 1/3$, $f(\text{'}l\text{'}) = 1/8$, $g(\text{'}l\text{'}) = 1/7$, $f(\text{'}d'\text{'}) = 3/4$, $g(\text{'}d'\text{'}) = 2/3$, $f(\text{'}l'\text{'}) = 7/8$, $g(\text{'}l'\text{'}) = 6/7$.

As we see, the propositions 'd' and 'l' satisfy the axiom A.4 so 'd' $\leqq$ 'l' which is equivalent to 'd' implies 'l', but such implication is physically completely unjustified. Now, using the definition of the orthogonality, we find that $d \leqq d'$; therefore, $d \perp d$.

Pool in his papers accepts the following definitions and axioms:

*Definition I.*1. An event-state structure is a triple (E, S, P):

(i) E is a set called the logic of the even-state structure and an element of E is called an event;
(ii) S is a set and an element of S is called a state;
(iii) P is a function $P: E \times S \to [0, 1]$ called the probability function and if $p \in E$ and $\alpha \in S$, then $P(p, \alpha)$ is called the probability of the occurrence of the event p in the state α;
(iv) if $p \in E$, then the subsets $S_1(p)$ and $S_0(p)$ of S are defined by

$$S_1(p) = \{\alpha \in S: P(p, \alpha) = 1\}$$
$$S_0(p) = \{\alpha \in S: P(p, \alpha) = 0\}.$$

*Axiom I.*3. If $p, q \in E$ and $S_1(p) \subset S_1(q)$, then $S_0(q) \subset S_0(p)$.

*Axiom I.*4. If $p \in E$, then there exists an event $p' \in E$ such that $S_1(p') = S_0(p)$ and $S_0(p') = S_1(p)$.

In our opinion, these axioms are not general enough. For example, if α are the states of ensembles consisting of beams, and p are the events of the type (transmission through the filter p), then the $P(p, \alpha)$ can be the transmission probabilities. In this case, the properties and the richness of the sets $S_1(p)$ and $S_0(p)$ depend on the beams and the filters in the particular experimental set-up and it is easy to give an example for which the Axioms I.3 and I.4 are not satisfied.

The above two examples support our thesis that it is extremely difficult, if not impossible, to axiomatise all possible experimental set-ups in the natural way.

Now we pass to the last section, where we formulate a programme of future investigations which could enable the answer of the title question of our paper.

4. *A Programme of Investigations*

We could not answer the title question of this paper, since we have been analysing some hypothetical general experimental set-ups. To answer the question whether the Hilbert space language is too rich to describe some physical phenomena, we should carefully analyse all real physical set-ups and observed phenomena, starting from solid-state physics and ending with high-energy elementary-particle physics. Such analysis should be done by physicists who really work in the specific branch of physics and who know all the subtleties of the experimental set-ups and of the theoretical analysis used to explain the data (to obtain the curves).

It is clear that it is quite difficult to find out that the language used is too rich; moreover, with the help of computers a beautiful agreement with the data can be obtained in most cases. However, one feature of the too rich language is the possibility of obtaining the same predictions using quite different models, which is equivalent to the lack of the unique theoretical explanation. The observation of such a situation can be a first hint for future investigations. In our opinion, one more or less sure method is to find such rigorously derived experimental predictions of a general nature, which can be verified in experiment, and to test them with full objectivity. In elementary-particle physics it can be the unitarity of the S matrix. The other method is to try to invent more economic language. In the discussion of the general experimental set-up, such possibilities were indicated. Especially interesting was that of Remark 1 where the notion of the transition probability disappears.

The other interesting problem is an operational status of quantum mechanics in its applications to many new phenomena. The operational status of quantum mechanics was discussed on the basis of experiments with polarised light and Stern–Gerlach experiments. Quantum mechanics as applied to high-energy elementary-particle scattering was not discussed in that context.

Another important problem is to investigate to what extent the good results which we obtain depend on all our particular assumptions and on the basic assumptions of the theory we used. Many models in elementary-particle physics are believed to be checked by the agreement of their predictions with experiment and are supposed to have a deeper physical meaning (not only to be a convenient parameterisation of the data). However, sometimes a careful analysis of the results shows that they are not deduced from the assumptions and they can only be rigorously derived from another set of assumptions which can have nothing in common with the physical ideas involved in the initial assumptions. To give an example, a careful analysis performed in the papers (Kielanowski & Kupczyński, 1971; Kupczyński, 1971) showed that the additivity assumption in the quark model applied with success for high-energy elementary-particle scattering can have nothing to do with the physical picture of a static quark model where the quarks are treated like hypothetical constituents of the elementary particles.

The programme which we have presented can be summarised as follows. Let us be more critical of the models we propose, of the conclusions we obtain, and let us check the operational status of the language we use to deal with data.

The investigation in this direction will be continued in the subsequent paper.

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